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Duality in gravity and higher spin gauge fields

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Abstract: Dual field theory realisations are given for linearised gravity in terms of gauge fields in exotic representations of the Lorentz group. The field equations and dual representations are discussed for a wide class of higher spin gauge fields. For non-linear Einstein gravity, such transformations can be implemented locally in light-cone gauge, or partially implemented in the presence of a Killing vector. Sources and the relation to Kaluza-Klein monopoles are discussed.

Keywords: Field Theories in Higher Dimensions, Duality in Gauge Field Theories, String Duality, Classical Theories of Gravity.

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Contents

1. Introduction

In D dimensions, the theory of a free abelian vector potential A_μ can be reformulated as a theory of a free n-form gauge field $\tilde{A}_{\mu_1 \ldots \mu_n}$ where $n = D - 3$. This gives two dual formulations of the same theory. In $D = 4$, both A and $\tilde{A}$ are 1-forms and the existence of two dual formulations leads to a symmetry of the equations of motion in which the field strength $F = dA$ transforms transforms into its Hodge dual, $*F$. This becomes part of a $\mathrm{SL}(2, \mathbb{R})$ symmetry of the equations of motion which acts on both fields and coupling constants, and for certain supersymmetric theories (such as those with $N = 4$ supersymmetry) this is broken to a discrete $\mathrm{SL}(2, \mathbb{Z})$ symmetry of the quantum theory.

These dualities extend to interacting theories in which A couples to other fields only through its field strength F. However, in the generalisation to non-abelian Yang-Mills theory, or to the minimal coupling to charged matter, the field equations and Bianchi identities involve the vector potential A explicitly, and there seems to be no local covariant way of implementing these duality transformations in classical field theory. In particular the $D = 4$ electromagnetic duality symmetries appear to be lost — the $N = 4$ super-Yang-Mills classical field equations only have an SL(2) symmetry in the abelian case.[1] Remarkably, there is considerable evidence that in $N = 4$ supersymmetric non-abelian Yang-Mills theories in four dimensions there is indeed such an $\mathrm{SL}(2,\mathbb{Z})$ S-duality symmetry of the full quantum theory even though there is no such symmetry of the classical non-abelian field equations. This duality symmetry has had profound implications for our understanding of the non-perturbative structure of these theories.

This S-duality symmetry has a geometric origin. In 6 dimensions, there are interacting (2,0) supersymmetric theories that reduce to theories containing super-Yang-Mills in $D < 6$. In the abelian case, the (2,0) theory is a theory of a 2-form gauge field with self-dual field strength, but the interacting theory cannot be a local covariant theory of interactions of such fields [3]. Reducing on a 2-torus gives a $D = 4$ theory with an $\mathrm{SL}(2,\mathbb{Z})$ S-duality symmetry arising from the diffeomorphisms of the 2-torus [4, 5]. This $D = 4$ theory reduces to super-Yang-Mills in a certain limit, but also inherits some of the features of the $D = 6$ theory that cannot be formulated in terms of a local covariant classical field theory.

In these examples, interacting classical field theory does not seem to provide a complete description of the full theory and in particular does not have the appropriate duality symmetries, and the study of the free limit and its symmetries turns out to be a surprisingly good guide to the properties of the full theory.

The purpose of this paper is to generalise such dualities to other types of gauge field, and in particular to the graviton. The starting point will be to generalise the duality transformations of the free vector field to dualities for free gravitons, governed by the linearised Einstein equations, resulting in dual formulations of linearised gravity in terms of exotic higher-rank gauge fields that were discussed in physical gauge in [6], and motivating the study of such higher-rank gauge fields. In $D = 4$, this leads to an SL(2) duality symmetry, just as for the spin one case.

These properties of the free graviton theory cannot be extended to give local covariant dual formulations of the generally covariant interacting field theory, just as the Maxwell dualities do not extend to classical Yang-Mills field theory. In the latter case, the Yang-Mills field theory does not give the full picture, and at least in certain supersymmetric theories, presumably should be replaced by some interacting theory which does have non-abelian dualities. It is possible that for gravity too, Einstein or supergravity field theories do not give the full picture and in fact arise only as a limit of some interacting theory which enjoys similar duality properties to the free theory. Some suggestions that this might be the case have arisen in recent work on string theory and M-theory [6, 7]. Such dualities would have many implications for

[1]In [1], a non-local construction of an electromagnetic dual of Yang-Mills theory was given, and in [2] it was shown that certain duality-invariant free actions do not admit local covariant non-abelian interactions.

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our understanding of gravity, and it seems worthwhile to explore this possibility and look for evidence in favour of it, or which could rule it out.

Such gravitational dualities should play a role in the interacting theory at least for spacetimes with isometries. Dimensionally reducing D-dimensional Einstein gravity on a circle gives an abelian vector gauge field A in $D-1$ dimensions, the graviphoton. This can be dualised to a $D-4$ form gauge field $\tilde{A}$ in $D-1$ dimensions, which couples to magnetically charged states arising from Kaluza-Klein monople solutions in D dimensions. It will be shown that this electromagnetic duality of the graviphoton can be formulated in terms of the D-dimensional gravitational field, with some of the dualities of the free theory extending to dualities of the interacting theory in the presence of a Killing vector, involving local covariant transformations of the curvature tensor. Dimensional reduction on a space with non-abelian isometry group, such as a sphere, gives rise to non-abelian gauge fields, and if there were non-abelian dualities of the type discussed above that involve these, these should extend to dualities of some of the components of the gravitational field that cannot be local and covariant.

2. Duality in physical gauge

Dualities are easily derived in physical (light-cone) gauge [6]. Introducing transverse coordinates $i, j = 1, \ldots, D-2$, a vector field A_μ has physical light-cone gauge degrees of freedom A_i in the vector representation of the little group $\mathrm{SO}(D-2)$ and can be dualised to an $n = D-3$ form

$$\tilde{A}_{j_1,\ldots,j_n} = \epsilon_{j_1,\ldots,j_n i} A^i \,. \tag{2.1}$$

The physical degrees of freedom are in equivalent representations of the little group and so A and $\tilde{A}$ give equivalent field theory representations of the physical degrees of freedom. Covariance suggests that the covariant field giving rise to the physical degrees of freedom $\tilde{A}_{j_1,\ldots,j_n}$ should be an n-form field $\tilde{A}_{\mu_1,\ldots,\mu_n}$ in D dimensions, and this should have gauge invariances sufficient to remove the unphysical degrees of freedom. Then $\tilde{A}$ should be an n-form gauge field, with gauge invariance $\delta\tilde{A} = d\lambda$ and field strength $\tilde{F} = d\tilde{A}$.

In the covariant theory, the duality relation can be recast as

$$\tilde{F} \equiv *F \tag{2.2}$$

which is a local covariant relation for the field strengths but not for the gauge potentials $A_\mu, \tilde{A}_{\mu_1\ldots\mu_n}$. The Maxwell equations in D dimensions for the 2-form field strength F with general sources are

$$dF = *\tilde{J}\,, \qquad d*F = *J \tag{2.3}$$

for electric and magnetic currents $J, \tilde{J}$ satisfying the conservation laws

$$d * J = 0\,, \qquad d * \tilde{J} = 0 \tag{2.4}$$

with a magnetic source $\tilde{J}$ for the 'Bianchi identity' $dF = 0$. These equations can be recast in a dual form in terms of the dual $(D-2)$-form field strength (2.2) as

$$d\tilde{F} = *J\,, \qquad d * \tilde{F} = *\tilde{J} \tag{2.5}$$

interchanging electric and magnetic currents and fields, and interchanging field equations and Bianchi identities. In regions in which $\tilde{J} = 0$, F can be solved for in terms of a potential A with $F = dA$, while in regions in which $J = 0$ one can solve for $\tilde{F} = d\tilde{A}$ in terms of a potential $\tilde{A}$.

In four dimensions, F and $\tilde{F}$ are both 2-forms so that the transformation $F \to *F$ preserves the form of the field equations and is the electromagnetic duality symmetry of the Maxwell equations. In certain theories, it is part of a larger SL(2) S-duality symmetry under which $(F, \tilde{F})$ transform as a doublet, as do the currents $(J, \tilde{J})$.

A physical gauge graviton in D dimensions is a transverse traceless tensor h_{ij} satisfying

$$h_{ij} = h_{ji}\,, \qquad h_i{}^i = 0\,. \tag{2.6}$$

One or both of the indices on h_{ij} can then be replaced by n anti-symmetric indices to give a dual form. Dualising one index gives a field

$$D_{i_1 \dots i_n\, k} = \epsilon_{i_1 \dots i_n l} h^l{}_k \tag{2.7}$$

so that (2.6) implies the conditions

$$D_{i_1 \dots i_n\, k} = D_{[i_1 \dots i_n]\, k}\,, \qquad D_{i_1 \dots i_{n-1} j}{}^j = 0\,, \qquad D_{[i_1 \dots i_n\, k]} = 0\,. \tag{2.8}$$

Dualising on both indices gives a field

$$C_{i_1 \dots i_n\, j_1 \dots j_n} = \epsilon_{i_1 \dots i_n m} \epsilon_{j_1 \dots j_n n} h^{mn} \tag{2.9}$$

which then satisfies

$$C_{i_1 \dots i_n j_1 \dots j_n} = C_{[i_1 \dots i_n][j_1 \dots j_n]} = C_{j_1 \dots j_n i_1 \dots i_n} \tag{2.10}$$

together with

$$C_{[i_1 \dots i_n j_1] j_2 \dots j_n} = 0, \qquad \delta^{i_n j_n} C_{i_1 \dots i_n j_1 \dots j_n} = 0\,. \tag{2.11}$$

The fields $D_{i_1 \dots i_n\, k}, C_{i_1 \dots i_n j_1 \dots j_n}$ should arise from covariant gauge fields $D_{\mu_1 \dots \mu_n\, \nu}$, $C_{\mu_1 \dots \mu_n \nu_1 \dots \nu_n}$ in D dimensions with gauge symmetries sufficient to remove all but the desired physical degrees of freedom. In the next section, these symmetries will be found and the covariant form of these duality transformations on the field strengths will be given.

These duality transformations can be generalised to the interacting theories in physical gauge. Yang-Mills theory can be formulated in the light-cone gauge in terms of a Lie algebra valued transverse vector potential A_i, and it is straightforward to make the substitution

$$A_i = \frac{1}{n!}\epsilon_{ij_1,\ldots,j_n}\tilde{A}^{j_1,\ldots,j_n} \tag{2.12}$$

to formulate the theory in terms of a Lie algebra valued transverse n-form $\tilde{A}$. As the substitution is local, the resulting theory is a *local* intreracting theory of $\tilde{A}$, although there are the usual light-cone gauge features such as inverse powers of the longitudinal momentum p^+. However, there is a problem in finding a covariant gauge theory that gives rise to this theory on going to physical gauge. The theory written in terms of A does arise from gauge-fixing the standard covariant Yang-Mills theory, but there is no local covariant classical field theory of a Lie algebra valued n-form gauge field with suitable gauge invariances that gives rise to this physical gauge theory of $\tilde{A}$ on gauge-fixing.

For gravity, the full interacting Einstein theory can be given in light-cone gauge as a theory of a symmetric transverse traceless tensor field h_{ij} with non-polynomial interactions [8]. (The field h_{ij} parameterises the coset space $\mathrm{SL}(D-2)/\,\mathrm{SO}(D-2)$, and this non-linear sigma-model structure is the origin of many of the non-polynomial interactions [8].) Here one can simply make the local substitution

$$h_{kl} = \frac{1}{n!}\epsilon^{i_1\ldots i_n}{}_l D_{i_1\ldots i_n\,k} \tag{2.13}$$

or

$$h_{mn} = \frac{1}{m!n!}\epsilon^{i_1\ldots i_n}{}_m \epsilon^{j_1\ldots j_n}{}_n C_{i_1\ldots i_n\,j_1\ldots j_n} \tag{2.14}$$

to obtain an interacting physical gauge theory of the dual potentials D or C, whose only non-localities are those involving inverse powers of the longitudinal momentum p^+. Again, while the theory written in terms of h_{ij} arises from the gauge fixing of Einstein's theory, the construction of a local covariant classical gauge theory of C or D that gives rise to these theories on gauge-fixing appears to be problematic.

In each case, there is no problem in dualising the interacting physical gauge theory and writing it as a local theory of dual potentials. In the free case, one can find covariant interacting theories that give rise to each of these dual forms, but in the interacting case, it appears that only some of these dual forms of the physical gauge theories arise from gauge fixing local covariant theories.

3. Linearised gravity

A free symmetric tensor gauge field $h_{\mu\nu}$ in D dimensions has the gauge symmetry

$$\delta h_{\mu\nu} = \partial_{(\mu}\xi_{\nu)} \tag{3.1}$$

and the invariant field strength is the linearised Riemann tensor

$$R_{\mu\nu\,\sigma\tau} = \partial_\mu \partial_\sigma h_{\nu\tau} + \ldots = -4\partial_{[\mu} h_{\nu][\sigma,\tau]}\,. \tag{3.2}$$

This satisfies

$$R_{\mu\nu\,\sigma\tau} = R_{\sigma\tau\,\mu\nu} \tag{3.3}$$

together with the first Bianchi identity

$$R_{[\mu\nu\,\sigma]\tau} = 0 \tag{3.4}$$

and the second Bianchi identity

$$\partial_{[\rho} R_{\mu\nu]\,\sigma\tau} = 0\,. \tag{3.5}$$

The natural free field equation in $D \geq 4$ is the linearised Einstein equation

$$R^{\sigma}{}_{\mu\,\sigma\nu} = 0 \tag{3.6}$$

which together with (3.5) implies

$$\partial^\mu R_{\mu\nu\,\sigma\tau} = 0\,, \tag{3.7}$$

where indices are raised and lowered with a flat background metric $\eta_{\mu\nu}$.

Dualising the linearised curvature gives tensors $S = *R$ and $G = *R*$ with components

$$S_{\mu_1\mu_2\ldots\mu_{n+1}\,\nu\rho} = \frac{1}{2}\epsilon_{\mu_1\mu_2\ldots\mu_{n+1}\alpha\beta} R^{\alpha\beta}{}_{\nu\rho} \tag{3.8}$$

and

$$G_{\mu_1\mu_2\ldots\mu_{n+1}\,\nu_1\nu_2\ldots\nu_{n+1}} = \frac{1}{4}\epsilon_{\mu_1\mu_2\ldots\mu_{n+1}\alpha\beta}\epsilon_{\nu_1\nu_2\ldots\nu_{n+1}\gamma\delta} R^{\alpha\beta\gamma\delta}\,, \tag{3.9}$$

where

$$n = D - 3\,. \tag{3.10}$$

The conditions (3.5)–(3.7) then become the equations for G given respectively by

$$G_{[\mu_1\mu_2\ldots\mu_{n+1}\,\nu_1]\nu_2\ldots\nu_{n+1}} = 0 \tag{3.11}$$

$$\partial^{\mu_1} G_{\mu_1\mu_2\ldots\mu_{n+1}\,\nu_1\nu_2\ldots\nu_{n+1}} = 0 \tag{3.12}$$

$$G_{\nu\mu_1\mu_2\ldots\mu_n\,\rho}{}^{\mu_1\mu_2\ldots\mu_n} = 0 \tag{3.13}$$

$$\partial_{[\rho} G_{\mu_1\mu_2\ldots\mu_{n+1}]\,\nu_1\nu_2\ldots\nu_{n+1}} = 0\,. \tag{3.14}$$

Here (3.11) can be regarded as a first Bianchi identity and (3.14) can be regarded as a second Bianchi identity, implying that G can be solved for in terms of a potential

$$C_{\mu_1\mu_2\ldots\mu_n\,\nu_1\nu_2\ldots\nu_n} = C_{[\mu_1\mu_2\ldots\mu_n]\,[\nu_1\nu_2\ldots\nu_n]} \tag{3.15}$$

satisfying

$$C_{[\mu_1\mu_2\ldots\mu_n\,\nu_1]\nu_2\ldots\nu_n} = 0 \tag{3.16}$$

by the expression

$$G_{\mu_1\mu_2\ldots\mu_{n+1}\,\nu_1\nu_2\ldots\nu_{n+1}} = \partial_{[\mu_1} C_{\mu_2\ldots\mu_{n+1}]\,[\nu_1\nu_2\ldots\nu_n,\nu_{n+1}]} \,. \tag{3.17}$$

The field strength is invariant under the gauge transformations

$$\delta C_{\mu\nu\ldots\kappa,\rho\sigma\ldots\ll} = \partial_{[\mu}\chi_{\nu\ldots\kappa]\rho\sigma\ldots\ll} + \partial_{[\rho}\chi_{\sigma\ldots\ll]\mu\nu\ldots\kappa} - 2\partial_{[\mu}\chi_{\nu\ldots\kappa\rho\sigma\ldots\ll]} \tag{3.18}$$

with parameter

$$\chi_{\mu_1\ldots\mu_{n-1}\nu_1\ldots\nu_n} = \chi_{[\mu_1\ldots\mu_{n-1}][\nu_1\ldots\nu_n]} \,. \tag{3.19}$$

Then (3.13) can be regarded as the field equation, implying (3.14).

Similarly, the conditions (3.5)–(3.7) become the equations for S given respectively by

$$S_{\mu_1\mu_2\ldots\mu_n\rho\,\nu}{}^{\rho} = 0 \tag{3.20}$$

$$\partial^\sigma S_{\sigma\mu_1\mu_2\ldots\mu_n\,\nu\rho} = 0\,, \qquad S_{\mu_1\mu_2\ldots\mu_{n+1}\,[\nu\rho,\sigma]} = 0 \tag{3.21}$$

$$S_{[\mu_1\mu_2\ldots\mu_{n+1}\,\nu]\rho} = 0\,, \qquad S_{\sigma[\mu_1\mu_2\ldots\mu_n\,\nu\rho]} = 0 \tag{3.22}$$

$$\partial_{[\sigma}S_{\mu_1\mu_2\ldots\mu_{n+1}]\,\nu\rho} = 0\,, \qquad \partial^\rho S_{\mu_1\mu_2\ldots\mu_{n+1}\,\nu\rho} = 0\,. \tag{3.23}$$

Here (3.22) can be regarded as first Bianchi identities and the first equation in (3.23) and the second equation in (3.20) can be regarded as second Bianchi identities, implying that S can be solved for in terms of a gauge field

$$D_{\mu_1\mu_2\ldots\mu_n\,\nu} = D_{[\mu_1\mu_2\ldots\mu_n]\,\nu} \tag{3.24}$$

satisfying

$$D_{[\mu_1\mu_2\ldots\mu_n\,\nu]} = 0 \tag{3.25}$$

with

$$S_{\mu\nu\ldots\rho\,\sigma\tau} = \partial_{[\mu}D_{\nu\ldots\rho]\,[\sigma,\tau]} \,. \tag{3.26}$$

The field strength S is then invariant under the gauge transformation

$$\delta D_{\mu\nu\ldots\sigma\,\rho} = \partial_{[\mu}\alpha_{\nu\ldots\sigma]\rho} + \partial_\rho\beta_{\mu\nu\ldots\sigma} - \partial_{[\rho}\beta_{\mu\nu\ldots\sigma]} \tag{3.27}$$

with parameters

$$\alpha_{\mu_1\ldots\mu_n\rho} = \alpha_{[\mu_1\ldots\mu_n]\rho}\,, \qquad \alpha_{[\mu_1\ldots\mu_n\rho]} = 0\,, \qquad \beta_{\mu\nu\ldots\sigma} = \beta_{[\mu\nu\ldots\sigma]}\,. \tag{3.28}$$

Then (3.20) can be taken as the field equation for the gauge field D, which then implies the first equation in (3.21) and the second in (3.23). Such gauge fields were first considered in [9], developing the discussion of massive gauge fields of [10], and further discussed in [11]–[13].

In $D = 5$, $C_{\mu\nu\,\rho\sigma}$ has the algebraic properties of the Riemann tensor; such gauge fields played a special role in [6] and similar gauge fields in $D = 4$ were considered in [14, 7]. In $D = 4$, all three dual gauge fields $h_{\mu\nu}, D_{\mu\nu}, C_{\mu\nu}$ are symmetric tensor gauge fields (linearised gravitons) with curvatures $R_{\mu\nu\rho\sigma}, S_{\mu\nu\rho\sigma}, G_{\mu\nu\rho\sigma}$. In $D = 3$, the Weyl tensor vanishes identically and the Riemann tensor is completely determined by the Ricci tensor, so that the field equation (3.6) implies that the field strength (3.2) vanishes and $h_{\mu\nu}$ is pure gauge. The simplest non-trivial linear field equation in $D = 3$ is

$$R^{\mu\nu}{}_{\mu\nu} = 0 \tag{3.29}$$

representing one degree of freedom. The curvature can then be dualised to give $G_{\mu\nu}$ ($G = *R*$) satisfying (3.11), (3.14), (3.12) but with 3.13 replaced by the field equation

$$G_\mu{}^\mu = 0 \tag{3.30}$$

which follows from (3.29). The potential C is a scalar and

$$G_{\mu\nu} = \partial_\mu \partial_\nu C \tag{3.31}$$

so that (3.30) implies the scalar satisfies the free scalar field equation

$$\partial^2 C = 0\,. \tag{3.32}$$

Similarly, the curvature can be dualised to $S_{\mu\,\nu_1\nu_2}$ ($S = *R$) satisfying (3.20), (3.23), (3.21) but with (3.22) replaced by

$$S_{[\mu\,\nu_1\nu_2]} = 0\,. \tag{3.33}$$

Then S can be solved for in terms of a vector potential D_μ with

$$S_{\mu\,\nu_1\nu_2} = \partial_\mu \partial_{[\nu_1} D_{\nu_2]} \tag{3.34}$$

and (3.20) is the Maxwell equation

$$\partial^\mu \partial_{[\mu} D_{\nu]} = 0\,. \tag{3.35}$$

The $D = 3$ dualilty betweeen a vector field D_μ and a scalar C is a well-known example of the electromagnetic duality of section 2, but here it is seen that they are also dual to a free graviton with the field equation (3.29). In $D = 2$, the Riemann tensor is completely determined by the Ricci scalar, so that the field equation (3.29) only has trivial solutions and there is no non-trivial linear field equation involving $h_{\mu\nu}$ alone.

4. General tensor gauge fields

It has been seen that a symmetric tensor gauge field can be dualised on one or both indices to get a new field representation of the same degrees of freedom. In this section, this will be generalised to other tensor gauge fields, which in principle can be dualised on any subset of their indices. The strategy is to start with a light-cone gauge potential, dualise to a potential in an equivalent representation of the little group, then seek a covariant formulation based on the dual potential. A general method for constructing a covariant field theory from any free light-cone gauge theory with a potential in any represetnation of the little group was given in [15].

Consider a gauge field with physical degrees of freedom

$$D_{i_1\ldots i_r\,j_1\ldots j_s} = D_{[i_1\ldots i_r]\,[j_1\ldots j_s]} \tag{4.1}$$

represented by a Young tableau with two columns, one of length r and one of length s. This satisfies

$$D_{[i_1\ldots i_r j_1]j_2\ldots j_s} = 0\,, \qquad D_{i_1\ldots i_{r-1}[i_r\,j_1\ldots j_s]} = 0\,, \qquad \delta^{i_r j_s} D_{i_1\ldots i_r\,j_1\ldots j_s} = 0\,. \tag{4.2}$$

It will be convenient to refer to an element of $\lll^r \otimes \lll^s \otimes \ldots \otimes \lll^t$ (where $\lll^r$ is the space of r-forms) represented by a Young tableau with columns of length $r, s, \ldots, t$ as a form of type $[r, s, \ldots, t]$, so that $D_{i_1\ldots i_r\,j_1\ldots j_s}$ is an $[r, s]$ form. This can be dualised on the first r indices to give a dual form of type $[\tilde{r}, s]$, on the second set of indices to give an $[r, \tilde{s}]$ form, or on both sets of indices to give an $[\tilde{r}, \tilde{s}]$ form, where $\tilde{r} = n+1-r$, $\tilde{s} = n+1-s$. (If $\tilde{r} < s$, it is conventional to take the longest column first, so that strictly speaking this is a tableau with the first column of length s and the second of length $\tilde{r}$. Such re-orderings are to be understood where necessary in what follows.)

For example, dualising on the first set gives

$$\tilde{D}_{k_1\ldots k_{\tilde{r}}\,j_1\ldots j_s} = \frac{1}{r!}\epsilon_{k_1\ldots k_{\tilde{r}}}{}^{i_1\ldots i_r} D_{i_1\ldots i_r\,j_1\ldots j_s}\,. \tag{4.3}$$

The physical degrees of freedom given by an $[r, s]$ form of $\mathrm{SO}(D-2)$ come from a covariant gauge field

$$D_{\mu_1\ldots\mu_r\,\nu_1\ldots\nu_s} = D_{[\mu_1\ldots\mu_r]\,[\nu_1\ldots\nu_s]} \tag{4.4}$$

which is an $[r, s]$ form of the Lorentz group $\mathrm{SO}(D-1, 1)$ satisfying

$$D_{[\mu_1\ldots\mu_r\,\nu_1]\nu_2\ldots\nu_s} = 0\,, \qquad D_{\mu_1\ldots\mu_{r-1}[\mu_r\,\nu_1\ldots\nu_s]} = 0\,. \tag{4.5}$$

The gauge transformations for such a gauge field are [12]

$$\delta D_{\mu_1\ldots\mu_r\,\nu_1\ldots\nu_s} = P_{r,s}\left[\partial_{[\mu_1}\alpha_{\mu_2\ldots\mu_r]\,\nu_1\ldots\nu_s} + \beta_{\mu_1\ldots\mu_r\,[\nu_1\ldots\nu_{s-1},\nu_s]}\right], \tag{4.6}$$

where α is a form of type $[r-1,s]$, β is a form of type $[r,s-1]$ and $P_{r,s}$ is the projector onto forms of type $[r,s]$ (Young symmetriser). These gauge transformations preserve the field strength

$$S_{\mu_1\dots\mu_{r+1}\,\nu_1\dots\nu_{s+1}} = -\partial_{[\mu_1}D_{\mu_2\dots\mu_{r+1}]\,[\nu_1\dots\nu_s,\nu_{s+1}]}\,. \tag{4.7}$$

The natural linear free field equations in $D \geq r+s+2$ are

$$S_{\mu_1\dots\mu_{r+1}\,\nu_1\dots\nu_{s+1}}\eta^{\mu_1\nu_1} = 0\,. \tag{4.8}$$

However, in dimension $D = r+s+1$, these field equations imply that the field strength vanishes identically, as was seen in particular examples in the previous section, and the simplest non-trivial linear field equation is

$$S_{\mu_1\dots\mu_{r+1}\,\nu_1\dots\nu_{s+1}}\eta^{\mu_1\nu_1}\eta^{\mu_2\nu_2} = 0 \tag{4.9}$$

with two contractions. In $D = r+s$, this equation only has trivial solutions, and the simplest non-trivial linear field equation is

$$S_{\mu_1\dots\mu_{r+1}\,\nu_1\dots\nu_{s+1}}\eta^{\mu_1\nu_1}\eta^{\mu_2\nu_2}\eta^{\mu_3\nu_3} = 0 \tag{4.10}$$

with three contractions. Similarly, the field equation in $D = r+s+2-p$ for $p \leq r, p \leq s$ is

$$S_{\mu_1\dots\mu_{r+1}\,\nu_1\dots\nu_{s+1}}\eta^{\mu_1\nu_1}\dots\eta^{\mu_p\nu_p} = 0 \tag{4.11}$$

with p contractions.

In the covariant gauge theory, the duality transformations are given in terms of the field strengths. An $[r,s]$ gauge field $D_{\mu_1\dots\mu_r\,\nu_1\dots\nu_s}$ is dualised to a $[\tilde{r},s]$ gauge field $\tilde{D}_{\mu_1\dots\mu_{\tilde{r}}\,\nu_1\dots\nu_s}$. The relation between the gauge fields is non-local, but the relation between the $[r+1,s+1]$ field strength S for D and the $[\tilde{r}+1,s+1]$ field strength $\tilde{S}$ for $\tilde{D}$ is local, $\tilde{S} = *S$. The field strengths for the $[\tilde{r},s]$, $[r,\tilde{s}]$ and $[\tilde{r},\tilde{s}]$ gauge fields dual to D are respectively $*S, S*$ and $*S*$.

This extends straightforwardly to gauge fields which are $[r,s,\dots,t]$ forms, which can be dualised on any set of anti-symmetric indices. For example, an $[r,s,t]$ form can be dualised to forms of types $[\tilde{r},s,t]$, $[r,\tilde{s},t]$, $[r,s,\tilde{t}]$, $[\tilde{r},\tilde{s},t]$, $[r,\tilde{s},\tilde{t}]$, $[\tilde{r},s,\tilde{t}]$, $[\tilde{r},\tilde{s},\tilde{t}]$. The field strength of a gauge field of type $[r_1,r_2,\dots,r_m]$ is given by acting on the gauge field with m derivatives and is a form of type $[r_1+1,r_2+1,\dots,r_m+1]$.

Consider next a 2-form gauge field $B_{\mu\nu}$ with gauge invariance $\delta B = d \ll$. This has physical degrees of freedom given by a transverse 2-form B_{ij}, which can be dualised in the usual way to give an $n-1$ form

$$\tilde{B}_{i_1\dots i_{n-1}} = \frac{1}{2}\epsilon_{i_1\dots i_{n-1}pq}B^{pq} \tag{4.12}$$

arising from gauge-fixing an $n-1$ form gauge field. Instead, one could attempt to dualise the anti-symmetric tensor in the same way as was done in the last section for a symmetric tensor. Dualising on one index gives a tensor

$$\hat{B}_{i_1\ldots i_n j} = \epsilon_{i_1\ldots i_n}{}^k B_{kj}\,. \tag{4.13}$$

However, the antisymmetry of B_{ij} implies that $\hat{B}$ is pure trace (i.e. the trace-free part vanishes) so that

$$\hat{B}_{i_1\ldots i_n j} = \tilde{B}_{[i_1\ldots i_{n-1}}\delta_{i_n]j} \tag{4.14}$$

and the usual dual $\tilde{B}$ defined by (4.12) is recovered. Similarly, dualising on both indices gives

$$\bar{B}_{i_1\ldots i_n j_1\ldots j_n} = \epsilon_{i_1\ldots i_n}{}^k \epsilon_{j_1\ldots j_n}{}^l B_{kl} \tag{4.15}$$

but this is necessarily of the form

$$\bar{B}_{i_1\ldots i_n j_1\ldots j_n} = \frac{1}{2}(\epsilon_{i_1\ldots i_n[j_1}\tilde{B}_{j_2\ldots j_n]} + \epsilon_{j_1\ldots j_n[i_1}\tilde{B}_{i_2\ldots i_n]}) \tag{4.16}$$

and again the usual dual $\tilde{B}$ is recovered. This generalises to an r form gauge field, which can be dualised in the usual way to an antisymmetric tensor gauge field of rank $\tilde{r} = n+1-r$, and again attempting to dualise on a subset of the indices leads back to the standard $\tilde{r}$-form dual.

A general second rank tensor k_{ij} can be decomposed into symmetric, anti-symmetric and trace parts

$$k_{ij} = h_{ij} + B_{ij} + \delta_{ij}\phi \tag{4.17}$$

arising from a graviton $h_{\mu\nu}$, a 2-form gauge field $B_{\mu\nu}$ and a scalar ϕ. This can be dualised on the first index to give a tensor

$$\hat{k}_{i_1\ldots i_n j} = \epsilon_{i_1\ldots i_n}{}^k k_{kj} \tag{4.18}$$

which is anti-symmetric on the first n indices $\hat{k}_{i_1\ldots i_n j} = \hat{k}_{[i_1\ldots i_n]j}$ but is otherwise arbitrary. This decomposes into

$$\hat{k}_{i_1\ldots i_n j} = D_{i_1\ldots i_n j} + \tilde{B}_{[i_1\ldots i_{n-1}}\delta_{i_n]j} + \epsilon_{i_1\ldots i_n j}\phi\,, \tag{4.19}$$

where $D_{i_1\ldots i_n j}$, $\tilde{B}_{i_1\ldots i_{n-1}}$ are the duals of h_{ij}, B_{ij} obtained above. Similarly, dualising on both indices gives

$$\bar{k}_{i_1\ldots i_n j_1\ldots j_n} = \epsilon_{i_1\ldots i_n}{}^k \epsilon_{j_1\ldots j_n}{}^l k_{kl} \tag{4.20}$$

which is given in terms of the duals $C_{i_1\ldots i_n j_1\ldots j_n}$, $\tilde{B}_{i_1\ldots i_{n-1}}$ as

$$\bar{k}_{i_1\ldots i_n j_1\ldots j_n} = C_{i_1\ldots i_n j_1\ldots j_n} + \frac{1}{2}(\epsilon_{i_1\ldots i_n[j_1}\tilde{B}_{j_2\ldots j_n]} + \epsilon_{j_1\ldots j_n[i_1}\tilde{B}_{i_2\ldots i_n]}) + \phi\epsilon_{i_1\ldots i_n k}\epsilon_{j_1\ldots j_n}{}^k\,. \tag{4.21}$$

The general situation can be summarised as follows. Consider some tensor gauge field $B_{\mu\nu\dots\rho}$ in D dimensions whose physical degrees of freedom are represented in light cone gauge by a field $B_{ij\dots k}$ in some tensor representation of the little group $\mathrm{SO}(D-2)$. Without loss of generality, attention can be restricted to irreducible representations, as the general case can be decomposed in terms of these, and each can be considered separately. Then any one of the indices $ij\dots k$ can be dualised to give $n = D-3$ antisymmetric indices and dual fields $\hat{B}_{i_1\dots i_n j\dots k} = \hat{B}_{[i_1\dots i_n]j\dots k}$, $\bar{B}_{ij_1\dots j_n\dots k} = \bar{B}_{i[j_1\dots j_n]\dots k}$ and so on. Any number of indices can be dualised simultaneously; dualising the first two indices, for example, will give a gauge field $b_{i_1\dots i_n j_1\dots j_n\dots k}$. Similarly, a set of r antisymmetrised indices can be dualised to a set of $\tilde{r} = n+1-r$ antisymmetrised indices. Not all the dual forms obtained in this way are independent. In general, the gauge field will be in an irreducible representation corresponding to a Young tableau with M columns of heights r_a, $a = 1,\dots,M$, and the independent dual representations are given by chosing any set of columns and replacing them by columns of dual height $\tilde{r}_a = n+1-r_a$. Finally, for each of the dual forms of the tensor field in physical gauge, one can construct a covariant gauge field and its gauge invariances, using the methods of [15], that would lead to that physical gauge field, and give a covariant form of the duality relations. The covariant gauge potential would typically be in the representation of $\mathrm{SO}(D-1,1)$ represented by a Young tableau with columns of the same lengths r_a as for the light-cone gauge potential, with the possibility of adding an arbitrary number of further columns, all of length $D-2$ [16]. The relations between the covariant gauge potentials is non-local, but there is a local relation between dual field strengths. Further details will be given elsewhere.

5. Sources

In Maxwell theory, one can introduce an electric source J_μ which couples to A_μ and an $n = D-3$ form magnetic current $J_{\mu_1\mu_2\dots\mu_n}$ which couples to the dual potential $\tilde{A}_{\mu_1\mu_2\dots\mu_n}$, with the field equations (2.3). In regions in which the magnetic current $\tilde{J}$ vanishes, F can be solved for in terms of a potential A, $F = dA$, while in regions in which the electric current J vanishes, $\tilde{F}$ can be solved for in terms of a dual potential $\tilde{A}$, $\tilde{F} = d\tilde{A}$.

In linearised gravity, one can consider adding sources $T_{\mu\nu}$, $U_{\mu_1\mu_2\dots\mu_n\,\nu}$ and $V_{\mu_1\mu_2\dots\mu_n\nu_1\nu_2\dots\nu_n}$ coupling naturally to the potentials $h_{\mu\nu}$, $D_{\mu_1\mu_2\dots\mu_n\,\nu}$ and $C_{\mu_1\mu_2\dots\mu_n\nu_1\nu_2\dots\nu_n}$, respectively. The linearised Einstein equations (for $D>2$) are

$$R^{\sigma}{}_{\mu\,\sigma\nu} = \bar{T}_{\mu\nu}\,, \tag{5.1}$$

where

$$\bar{T}_{\mu\nu} = T_{\mu\nu} + \frac{1}{D-2}\eta_{\mu\nu}T \tag{5.2}$$

and $T_{\mu\nu}$ is the energy-momentum tensor, the trace term being added so that the Bianchi identities imply that $T_{\mu\nu}$ is identically conserved, $\partial^\mu T_{\mu\nu} = 0$. Similarly, the natural field equations with the source U is

$$S_{\mu_1\mu_2\ldots\mu_n\rho\,\nu}{}^{\rho} = \bar{U}_{\mu_1\mu_2\ldots\mu_n\,\nu}\,, \tag{5.3}$$

where

$$\bar{U}_{\mu_1\mu_2\ldots\mu_n\,\nu} = U_{\mu_1\mu_2\ldots\mu_n\,\nu} + \frac{n}{2}\eta_{\nu[\mu_1}U_{\mu_2\mu_3\ldots\mu_n]\rho}{}^{\rho} \tag{5.4}$$

so that the Bianchi identities for S imply that U is conserved,

$$\partial^{\mu_1}U_{\mu_1\mu_2\ldots\mu_n\,\nu} = 0\,, \qquad \partial^{\nu}U_{\mu_1\mu_2\ldots\mu_n\,\nu} = 0\,. \tag{5.5}$$

The field equation (5.3) then implies that U is a source for the gravitational Bianchi identity (3.4):

$$R_{[\mu\nu\,\sigma]\tau} = \frac{1}{n!}\epsilon_{\mu\nu\,\sigma}{}^{\mu_1\mu_2\ldots\mu_n}\bar{U}_{\mu_1\mu_2\ldots\mu_n\,\tau} \tag{5.6}$$

Similarly, the stress-energy tensor $T_{\mu\nu}$ is a 'source' on the right hand side of the Bianchi identity (3.22). In a region in which U vanishes, the Bianchi identity holds and the gravitational field can be written in terms of $h_{\mu\nu}$ satisfying the field equation (5.1), while in a region in which T vanishes, the Bianchi identities (3.22) hold and the linearised gravitational field can be expressed in terms of the dual field $D_{\mu\ldots\nu}$ with the field equation (5.3).

Including a source for the field equation (3.13) gives

$$G_{\nu\mu_1\mu_2\ldots\mu_n\,\rho}{}^{\mu_1\mu_2\ldots\mu_n} = \bar{V}_{\nu\rho}\,. \tag{5.7}$$

However, it follows from (3.9), (5.1) that $\bar{V}_{\nu\rho}$ is related to the usual energy-momentum tensor,

$$\bar{V}_{\nu\rho} = aT_{\nu\rho} + b\eta_{\nu\rho}T \tag{5.8}$$

for some n-dependent coefficients a, b. A source for (3.11) of the form

$$G_{[\mu_1\mu_2\ldots\mu_{n+1}\,\mu_{n+2}]\nu_1\ldots\nu_n} = W_{[\mu_1\mu_2\ldots\mu_{n+1}\,\mu_{n+2}]\nu_1\ldots\nu_n} \tag{5.9}$$

can be given in terms of U, so that

$$G_{[\mu_1\mu_2\ldots\mu_{n+1}\,\mu_{n+2}]\nu_1\ldots\nu_n} = \epsilon_{\mu_1\mu_2\ldots\mu_{n+1}\,\mu_{n+2}}{}^{\rho}\bar{U}_{\nu_1\ldots\nu_n\,\rho}\,. \tag{5.10}$$

Thus, although linearised gravity can be represented in terms of three different types of fields, there are only two independent types of source that can arise, T and U.

Consider then linearised gravity, formulated in terms of a field strength $R_{\mu\nu\rho\sigma}$, satisfying these generalised field equations and 'Bianchi identities' with sources T, U. If there is no magnetic source, $U = 0$, the standard Bianchi identities hold and the field strength can be solved for in terms of the gauge field $h_{\mu\nu}$ as usual, or in terms of the double-dual potential C. However, in a region in which $U \neq 0$ but $T = 0$, one can instead solve for R in terms of the dual potential D and the theory has to be formulated in terms of this dual potential. Outside regions in which $U \neq 0$, one can find a linearised metric $h_{\mu\nu}$ locally, but the presence of regions with non-zero U will often lead to Dirac string singularities in $h_{\mu\nu}$. The situation is similar for the more general fields considered in section 4.

Some of the possible physical consequences of a magnetic source for gravity were discussed in [18]. It seems unlikely that there is such 'magnetic energy' in the observable universe. However, while it is expected that there are very few magnetic monopoles in the observable universe, or perhaps none at all, magnetic monopoles have come to play a central role in our theoretical understanding of the non-perturbative structure of many gauge theories, and it is conceivable that magnetic sources for gravity could play a similarly important role in non-perturbative gravity, despite their apparent absence. As will be seen, there is a sense in which Kaluza-Klein monopoles can be regarded as such magnetic sources.

6. Spacetimes with Killing vectors

In this section, the possibility of performing duality transformations in the non-linear Yang-Mills or Einstein equations will be considered in the special cases of configurations with an extra symmetry.

In Yang-Mills theory with Lie-algebra valued A and $F = dA + A \wedge A$, the field equations $D * F = 0$ and Bianchi identities $DF = 0$ involve the background covariant derivative $D\phi = d\phi + [A, \phi]$ and the presence of the explicit gauge potential A means that there is no local covariant generalisation of electromagnetic duality in general. However, consider a gauge-field configuration that admits a 'Killing scalar' α i.e. a gauge field configuration A that is invariant under the gauge transformation $\delta A = D\alpha$ for some parameter α, so that α must be covariantly constant

$$D\alpha = 0$$

implying $[F, \alpha] = 0$. Then the 2-form $f = tr(\alpha F)$ satisfies the Maxwell equations

$$df = 0\,, \qquad d * f = 0$$

and these can be re-expressed in terms of a dual field strength $\tilde{f} = *f$. Furthermore, f can be solved for in terms of a potential a, $f = da$, and similarly $\tilde{f}$ can be expressed in terms of a dual potential, $\tilde{f} = d\tilde{a}$. Then the existence of a Killing scalar allows linearisation of the field equations for the components f of F and this subset of the field equations can be dualised.

Consider a D-dimensional spacetime with metric $g_{\mu\nu}$ and a Killing vector k^μ satisfying the Killing equation

$$\nabla_{(\mu} k_{\nu)} = 0\,. \tag{6.1}$$

Then the Killing equation and Ricci identities imply that the (full nonlinear) Riemann curvature tensor satisfies

$$R_{\mu\nu\rho\sigma}k^{\sigma} = \nabla_{\rho} f_{\mu\nu} \,,$$

where

$$f_{\mu\nu} = 2\partial_{[\mu} k_{\nu]} \,.$$

The first Bianchi identity implies

$$df = 0$$

while the (full nonlinear) Einstein equation

$$R_{\mu\nu} = \bar{T}_{\mu\nu}, \qquad \bar{T}_{\mu\nu} = T_{\mu\nu} - \frac{1}{D-2} T g_{\mu\nu}$$

(where $T = g^{\mu\nu}T_{\mu\nu}$) implies the Maxwell equation

$$d * f = *J \,,$$

where the current is

$$J_{\mu} = \bar{T}_{\mu\nu} k^{\nu} \,.$$

If $T \neq 0$, this differs from the conserved momentum

$$P_{\mu} = T_{\mu\nu} k^{\nu}$$

(the conservation of which is a consequence of the isometry symmetry) by

$$J_{\mu} - P_{\mu} = \frac{1}{2} k_{\mu} R$$

which is automatically conserved as (6.1) implies $\nabla^{\mu}(k_{\mu}R) = 0$. Then the presence of a Killing vector has allowed some of Einstein's equations – those with at least one component in the Killing direction – to be rewritten as the free Maxwell equations for the field stength $f = da$ with 'gauge potential' $a_{\mu} = g_{\mu\nu}k^{\nu}$.

A magnetic source $\tilde{J}$ for f

$$df = *\tilde{J}$$

would correspond to a source for the 1st Bianchi identity

$$R_{[\mu\nu\rho]\sigma}k^{\sigma} = (*\tilde{J})_{\mu\nu\rho}$$

and the presence of such sources would mean that f couldn't be solved for in terms of a local potential a_{μ} and so the curvature couldn't be expressed in terms of a metric $g_{\mu\nu}$ alone — with magnetic monopole sources for a, the potential a would have Dirac string singularities and so the metric would also have these; such Dirac string singularities in the metric are sometimes referred to as Misner strings.

In the non-linear theory, one can define a tensor $S = *R$, as in (3.8). The Bianchi identities (3.21), (3.23) become

$$\nabla^{\sigma} S_{\sigma\mu_1\mu_2\ldots\mu_n\,\nu\rho} = 0\,, \qquad S_{\mu_1\mu_2\ldots\mu_{n+1}\,[\nu\rho;\sigma]} = 0\,, \tag{6.2}$$

$$\nabla_{[\sigma} S_{\mu_1\mu_2\ldots\mu_{n+1}]\,\nu\rho} = 0\,, \qquad \nabla^{\rho} S_{\mu_1\mu_2\ldots\mu_{n+1}\,\nu\rho} = 0 \tag{6.3}$$

with covariant derivatives involving the Christoffel connection. Thus the metric appears explicitly, and the theory cannot be written in terms of the dual potential D alone. If there is a magnetic source, the field equation will be (5.3). With a Killing vector k, $\tilde{f} = *f$ satisfies

$$\nabla_{\rho} \tilde{f}_{\mu_1\mu_2\ldots\mu_{n+1}} = S_{\mu_1\mu_2\ldots\mu_{n+1}\,\rho\sigma} k^{\sigma}\,. \tag{6.4}$$

In the absence of matter, $T_{\mu\nu} = 0$, and S will satisfy (3.22), so that

$$d\tilde{f} = 0 \tag{6.5}$$

and locally there is a dual potential n-form $\tilde{a}$ such that

$$\tilde{f} = d\tilde{a}\,.$$

This satisfies the field equation

$$\nabla^{\rho} \tilde{f}_{\mu_1\mu_2\ldots\mu_n\rho} = \tilde{J}_{\mu_1\mu_2\ldots\mu_n}\,,$$

where

$$\tilde{J}_{\mu_1\mu_2\ldots\mu_n} = \bar{U}_{\mu_1\mu_2\ldots\mu_n\,\sigma} k^{\sigma}\,.$$

Thus the presence of a Killing vector allows the rewriting of some of Einstein's equations in terms of the dual potential $\tilde{a}$. In the linearised theory, this dual potential is given in terms of the dual gravitational field D by

$$\tilde{a}_{\mu_1\mu_2\ldots\mu_n} = D_{\mu_1\mu_2\ldots\mu_n\,\sigma} k^{\sigma}\,.$$

On dimensional reduction from D to $D-1$ dimensions on a circle, the metric gives rise to an abelian gauge field A_m in $D-1$ dimensions, together with a scalar field V and a metric g_{mn} throught the ansatz

$$ds^2 = V(dy + A_m dx^m)^2 + V^{-1} g_{mn} dx^m dx^n\,, \tag{6.6}$$

where V, A_m and g_{mn} depend on the coordinates x^m, so that $a_{\mu} = (V, VA_m)$. The above equations give rise to standard Kaluza-Klein field equations for V, A_m and g_{mn}. The vector field A_m is defined up to a gauge transformation

$$A_{\mu} \to A_m + \partial_m \rho \tag{6.7}$$

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as such a change can be absorbed into the coordinate transformation

$$y \to y - \rho\,.$$

The invariant field-strength

$$F_{mn} = \partial_{[m}A_{n]} \tag{6.8}$$

is the twist or vorticity of the vector field k. If the potential A has a Dirac string singularity, then the metric has a Misner string singularity.

7. Kaluza-Klein monopoles and dimensional reduction

A vector field A_m in d dimensions is associated with electrically charged 0-branes, which couple to A_m, and with magnetically charged $d-4$ branes which couple to the dual potential, which is a $d-3$ form $\tilde{A}_{m_1\ldots m_{d-3}}$. Suppose that the vector field arises from the Kaluza-Klein reduction of gravity from D to $d = D-1$ dimensions on a circle, with $D \geq 5$. The D-dimensional metric $g_{\mu\nu}$ of the form (6.6) gives a metric g_{mn}, a vector field A_m and a scalar ϕ in $D-1$ dimensions. Then states which are electrically charged in d-dimensions arise from states carrying momentum in the (internal) y direction, with the D-momentum P^μ giving the $D-1$-momentum P^m and the electric charge $Q = P^D$ associated with A on dimensional reduction, $P^\mu = (P^m, Q)$. In particular, the 0-branes arise from such momentum modes. For configurations in d dimensions with magnetic charge, A will in general have Dirac string singularities. These can be avoided by modifying the topology and regarding the potential as a connection for a non-trivial bundle, or (in the absence of electric charge) by going to the dual formulation, with the magnetic charges acting as sources for a non-trivial dual potential $\tilde{A}$. The Dirac string singularities in A give rise to Misner string singularities in the D dimensional metric $g_{\mu\nu}$. However, in some cases these can be eliminated by modifying the topology of the D dimensional space time so that it is not a product of a circle with some space, but is instead a circle bundle. In particular, the magnetically charged $d-4$ branes arise from Kaluza-Klein monopole solutions in D dimensions of the form $N_4 \times \mathbb{R}^{D-5,1}$, where N_4 is euclidean Taub-NUT space, and the reduction is over the S^1 fibre of N_4. For example, the D6-brane of type-IIA string theory couples to a RR 7-form potential, the dual of the RR vector field, and arises from the KK monopole solution of M-theory in $D = 11$. Alternatively, one could attempt to eliminate the string singularities by formulating the theory in terms of a dual potential, as will now be discussed.

Consider first linearised gravity, with the D dimensional graviton $h_{\mu\nu}$ giving a graviton h_{mn}, a vector field $A_m = h_{my}$ and a scalar $\phi = h_{yy}$ in $d = D-1$ dimensions. In D dimensions, the graviton can be dualised to a gauge field $D_{\mu_1\ldots\mu_{D-3}\,\nu}$, while in d dimensions, h_{mn} can be dualised to a gauge field $d_{m_1\ldots m_{d-3}\,n}$, A_m can be dualised to a $d-3$ form gauge field $\tilde{A}_{m_1\ldots m_{d-3}}$ and ϕ can be dualised to a $d-2$ form gauge field $\tilde{\phi}_{m_1 \ldots m_{d-2}}$. On dimensional reduction, $D_{\mu_1 \ldots \mu_{D-3}\,\nu}$ gives rise to the dual graviton $d_{m_1 \ldots m_{d-3}\,n} = D_{m_1 \ldots m_{d-3} y\, n}$, the dual scalar $\tilde{\phi}_{m_1 \ldots m_{d-2}} = D_{m_1 \ldots m_{D-3}\, y}$ and the dual vector $\tilde{A}_{m_1 \ldots m_{d-3}} = D_{m_1 \ldots m_{d-3} y\, y}$ (the remaining components $D_{m_1 \ldots m_{D-3}\, n}$ give no further independent degrees of freedom). Thus in the linearised theory, magnetically charged branes in d dimensions acting as sources of the gauge field $\tilde{A}$ lift to configurations that are sources of the components $D_{m_1 \ldots m_{d-3} y\, y}$ of the dual graviton and formulating in terms of $\tilde{A}$ or D avoids the string singularities that would otherwise arise for h or A. Thus Kaluza-Klein monopoles naturally couple to the dual graviton D, and so can be thought of as carrying the 'magnetic energy-momentum' current U.

If a BPS magnetic $d-4$ brane in d dimensions is wrapped on a rectangular $d-4$ torus, then the resulting 0-brane in 4 dimensions has mass proportional to

$$\tilde{A}_{0 m_1 \ldots m_{d-4}} R^{m_1} R^{m_2} \cdots R^{m_{d-4}} \,,$$

where R^m is the radius of the circle in the x^m direction. This lifts to the following expression in D dimensions

$$D_{0 \mu_1 \ldots \mu_{D-4} y\, y} R^{\mu_1} R^{\mu_2} \cdots R^{\mu_{d-4}} R^y R^y$$

with the correct quadratic dependence on the radius R^y of the extra circle.

In the non-linear theory, we have seen that in the presence of a Killing vector, it is possible to dualise some of the gravitational degrees of freedom, and in particular, it is possible to construct in this way the dual gravitational potential to which the Kaluza-Klein monopoles couple.

8. Duality symmetries in four dimensions

8.1 Maxwell theory in four dimensions

In four euclidean dimensions, one can consistently impose the self-duality condition $F = *F$ so that the Bianchi identity $dF = 0$ implies the field equation $d * F = 0$, but is stronger. In 3+1 dimensions this is inconsistent as $(*)^2 = -1$. However, given two 2-form field strengths $F_i = (F, \tilde{F})$ $(i = 1, 2)$ one can impose

$$F_i = J_i{}^j * F_j \tag{8.1}$$

for any matrix $J_i{}^j$ that satisfies $J^2 = -1$. Then the Bianchi identities $dF_i = 0$ imply the field equations $d(J_i{}^j * F_j) = 0$, and this is the form of the field equations proposed in [17]. With the natural choice

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{8.2}$$

this gives $\tilde{F} = *F$, $F = - * \tilde{F}$. More generally, different choices of field-dependent J lead to different field equations and hence define different theories.

The equations (8.1) are invariant under an $\mathrm{SL}(2,\mathbb{R})$ symmetry, with F_i transforming as

$$F \to SF\,, \tag{8.3}$$

where $S_i{}^j$ is an $\mathrm{SL}(2,\mathbb{R})$ matrix provided J transforms as

$$J \to SJS^{-1}\,. \tag{8.4}$$

For the standard theory with coupling constant g and and a theta-angle θ, a J with these transformation properties can be constructed as follows. Combining the coupling constants into the complex modulus

$$\tau = \tau_1 + i\tau_2 = \theta + \frac{i}{g^2} \tag{8.5}$$

then the flat 2-metric γ_{ij} given by

$$\gamma_{ij} = \frac{V}{\tau_2}\begin{pmatrix} 1 & \tau_1 \\ \tau_1 & |\tau|^2 \end{pmatrix} \tag{8.6}$$

transforms as

$$\gamma \to S\gamma S^t \tag{8.7}$$

provided τ transforms under SL(2) through fractional linear transformations and the volume V is invariant. Then introducing the alternating tensor ϵ_{ij} with components $\epsilon_{12} = \sqrt{\gamma}$, $\gamma = det\gamma_{ij}$, a J transforming as (8.4) and satisfying $J^2 = -1$ can be defined as

$$J_i{}^j = \frac{1}{\sqrt{\gamma}}\gamma_{ik}\epsilon^{kj}\,. \tag{8.8}$$

This is the well-known $\mathrm{SL}(2,\mathbb{R})$ duality symmetry of the classical theory which, in the quantum theory with $N = 4$ supersymmetry, is broken to a discrete $\mathrm{SL}(2,\mathbb{Z})$ subgroup. The 2-metric can be interpreted as the metric on a 2-torus, with the $\mathrm{SL}(2,\mathbb{Z})$ acting as large diffeomorphisms on the 2-torus.

This can be generalised to certain theories of abelian gauge fields interacting with scalars, such as those that occur in the bosonic sector of ungauged supergravity theories, with m gauge fields F_a $(a = 1,\ldots,m)$ whose field equations can be written as $d\tilde{F}_a = 0$, where $\tilde{F}_a$ is given in terms of $F_a = dA_a$ in terms of the variation of the bosonic action S by

$$\tilde{F}_a \equiv *(\delta S/\delta F_a) = *F_a + \cdots\,. \tag{8.9}$$

The $2m$ field strengths $F_i = (F_a, \tilde{F}_a)$ are combined into a $2m$-vector F_i and a scalar-dependent J is determined by requiring that $J^2 = -1$ and that (8.1) implies (8.9). The equations will then be invariant under some duality group G for which F_i transforms as a $2m$-dimensional representation and the transformations of the scalars are such that J transforms in the appropriate way. For example, in $N = 4$ supergravity theories, each vector field has field strengths fitting into an SL(2) doublet, with a J constructed as in (8.8) from a complex scalar field τ taking values in the coset space $\mathrm{SL}(2,\mathbb{R})/\,\mathrm{U}(1)$ and transforming as in (8.4).

8.2 Linearised gravity in four dimensions

Gravitational analogues of electromagnetic duality have been considered in four-dimensional higher derivative theories of gravity [19], and in other gravitational theories in [20], but here attention will be restricted to the linearised Einstein theory. In four euclidean dimensions, one can impose the self-duality condition

$$R_{\mu\nu\,\sigma\tau} = \frac{1}{2}\epsilon_{\mu\nu\rho\kappa}R^{\rho\kappa}{}_{\rho\sigma}$$

or $R = *R$ so that the Bianchi identity (3.4) implies the field equations (3.6), but with Lorentzian signature this would imply zero curvature. In 3+1 dimensions, the dual potentials $D_{\mu\nu}$ and $C_{\mu\nu}$ are again symmetric tensor gauge fields and one can seek a generalisation of the construction used above for the Maxwell field. The curvature tensor $R_{\mu\nu\rho\sigma}$ can be dualised on the first two indices to give $*R$, on the second two to give $R*$ or both to give $*R*$. These can be formed into a 2×2 matrix of tensors $\bar{R}_{\mu\nu\sigma\tau\,ij}$ $(i,j=1,2)$ given by

$$\bar{R}_{ij} = \begin{pmatrix} R & *R \\ R* & *R* \end{pmatrix} \tag{8.10}$$

satisfying

$$\bar{R}_{\mu\nu\sigma\tau\,ij} = \bar{R}_{\sigma\tau\mu\nu\,ji}$$

together with the constraint

$$\bar{R}_{\mu\nu\sigma\tau\,ij} = J_i{}^k(*\bar{R})_{\mu\nu\sigma\tau\,kj} \tag{8.11}$$

with J given by (8.2).

This can be generalised to consider tensors $\bar{R}_{\mu\nu\sigma\tau\,ij}$ satisfying (8.11) and allow J to be any matrix satisfying $J^2=-1$, with different choices determining different interactions, so that (8.10) is generalised. (The choice (8.2) recovers (8.10).) Then if the curvatures satisfy the first Bianchi identities

$$\bar{R}_{[\mu\nu\sigma]\tau\,ij} = 0 \tag{8.12}$$

the constraint (8.11) implies the field equations

$$\eta^{\nu\tau}\bar{R}_{\mu\nu\sigma\tau\,ij} = 0 \tag{8.13}$$

and

$$\bar{R}_{\mu\nu\sigma\tau\,ij} = \bar{R}_{\sigma\tau\mu\nu\,ij}$$

and

$$\bar{R}_{\mu\nu\sigma\tau\,ij} = \bar{R}_{\mu\nu\sigma\tau\,ji}$$

so that R_{ij} is a symmetric 2×2 matrix.

A particularly interesting choice of J is that given by (8.8), so that one is introducing a gravitational 'theta-angle' as well as the usual gravitational coupling constant. This is the system that arises from the dimensional reduction of the free (4,0) conformal multiplet in 5+1 dimensions [7]. Then (8.11) has an SL(2) symmetry under which $\bar{R}_{\mu\nu\sigma\tau\, ij}$ transforms as a 2nd rank tensor

$$\bar{R}_{\mu\nu\sigma\tau} \to S\bar{R}_{\mu\nu\sigma\tau}S^t \tag{8.14}$$

(which is a symmetric tensor if (8.12) holds) while J transforms as (8.4). This SL(2) is a gravitational analogue of the S-duality of the electromagnetic field and mixes the field equations (8.13) with the Bianchi identities (8.12). Moreover, (8.11) then implies the SL(2)-invariant constraint

$$\gamma^{ij}\bar{R}_{\mu\nu\sigma\tau\, ij} = 0 \tag{8.15}$$

so that R_{ij} is a symmetric matrix with vanishing trace(8.15).

Writing the components of R_{ij} as in [7] as

$$\bar{R}_{\mu\nu\sigma\tau\, ij} = \begin{pmatrix} R_{\mu\nu\sigma\tau} & \tilde{R}_{\mu\nu\sigma\tau} \\ \tilde{R}_{\mu\nu\sigma\tau} & \hat{R}_{\mu\nu\sigma\tau} \end{pmatrix} = \begin{pmatrix} R_{\mu\nu\sigma\tau} & S_{\sigma\tau\mu\nu} \\ S_{\mu\nu\sigma\tau} & G_{\mu\nu\sigma\tau} \end{pmatrix}$$

with J given by (8.8), the Bianchi identities imply that these are the curvature tensors for the gravitons

$$\bar{h}_{\mu\nu\, ij} = \begin{pmatrix} h_{\mu\nu} & \tilde{h}_{\mu\nu} \\ \tilde{h}_{\mu\nu} & \hat{h}_{\mu\nu} \end{pmatrix} = \begin{pmatrix} h_{\mu\nu} & D_{\mu\nu} \\ D_{\mu\nu} & C_{\mu\nu} \end{pmatrix}.$$

Then (8.15) implies

$$\hat{R}_{mnpq} - 2\tau_1\tilde{R}_{mnpq} + |\tau|^2 R_{mnpq} = 0 \tag{8.16}$$

so that

$$\hat{h}_{mn} - 2\tau_1\tilde{h}_{mn} + |\tau|^2 h_{mn} = 0$$

up to gauge transformations.

The duality constraint (8.11) gives the following relations between the curvature tensors (suppressing the spacetime indices)

$$R = \frac{1}{\tau_2}(- * \tilde{R} + \tau_1 * R)\tilde{R} = \frac{1}{\tau_2}(- * \hat{R} + \tau_1 * \tilde{R})\hat{R} = \frac{1}{\tau_2}(|\tau|^2 * \tilde{R} - \tau_1 * \hat{R})\,. \tag{8.17}$$

This implies that $\tilde{R}, \hat{R}$ are given in terms of R by

$$\tilde{R} = \frac{1}{g^2} * R - \theta R\hat{R} = 2\tau_1\tau_2 * R - (\tau_1^2 - \tau_2^2)R \tag{8.18}$$

so that only one of the three gravitons $h, \tilde{h}, \hat{h}$ is independent. The one remaining independent graviton can be taken to be h with action

$$S = \frac{1}{2l^2}\int d^4x\, (\sqrt{g}R)_{\rm quad}\,,$$

where the Planck length is given by

$$l = \sqrt{V} g^2$$

and $(\sqrt{g}R)_{\rm quad}$ is the Einstein action truncated to terms quardatic in h_{mn}. There are two dimensionless coupling constants, g, θ. While g can be absorbed into the the gravitational coupling l, there is the interesting possibility of introducing a gravitational θ-parameter.

The duality transformations (8.14) take R to linear combinations of $R, \tilde{R}, \hat{R}$ which through (8.18) can be written as linear combinations of $R, *R$. The action of the $\mathrm{SL}(2,\mathbb{Z})$ element

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

is to take

$$\tau \to -\frac{1}{\tau}$$

and

$$F \to \tilde{F}, \qquad \tilde{F}_2 \to -F$$

while

$$R \to \hat{R}, \qquad \hat{R} \to R, \qquad \tilde{R} \to -\tilde{R}\,.$$

Note that this preserves the constraint (8.16). For self-dual F_i satisfying (8.1), the transformation is the standard duality transformation

$$F \to \frac{1}{g^2} * F + \theta F$$

while for self-dual $\bar{R}$ satisfying (8.17)

$$R_{mnpq} \to 2\tau_1\tau_2(*R)_{mnpq} - (\tau_1^2 - \tau_2^2) R_{mnpq}\,.$$

For $\theta = 0$, this takes $g \to 1/g$ and so relates strong and weak coupling regimes. Both the linear gravity and Maxwell theory are self-dual, with the strong coupling regime described by the same theory.

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